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Journal of Computational Physics 217 (2006) 159–165
JOURNAL OF COMPUTATIONAL PHYSICS
www.elsevier.com/locate/jcp

On upscaling operator-stable Lévy motions in fractal porous media

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Received 10 August 2005; received in revised form 21 December 2005; accepted 19 January 2006
Available online 28 February 2006

Abstract

The dynamics of motile particles, such as microbes, in random porous media are modeled with a hierarchical set of stochastic differential equations which correspond to micro, meso and macro scales. On the microscale the motile particle is modeled as an operator stable Lévy process with stationary, ergodic, Markov drift. The micro to meso and meso to macro scale homogenization is handled with generalized central limit theorems. On the mesoscale the Lagrangian drift (or the Lagrangian acceleration) is assumed Lévy to account for the fractal character of many natural porous systems. Diffusion on the mesoscale is a result of the microscale asymptotics while diffusion on the macroscale results from the mesoscale asymptotics. Renormalized Fokker–Planck equations with time dependent dispersion tensors and fractional derivatives are presented at the macro scale.

Keywords: Operator stable Lévy; Porous media; Upscaling; Random; Fractal

1. Introduction

Self-motile colloidal-sized particles, such as microbes, play a fundamental role in the transport of constituents and fluids in natural porous media. The ability to swim allows a microbe to move in a preferential direction toward desirable energy sources (light, chemical, magnetic, thermal, etc.) [2]. Randomly swimming microbes can often be modeled as α-stable Lévy motions [11]. Using the concept of directional preference put forth in [8], swimming in a preferential direction based on an energy gradient can be modeled using an operator-stable Lévy motion.

Uncertainty in microbial dynamics is manifest on several scales within porous media. We attempt to quantify this uncertainty on three scales via use of renormalizing central limit theorems applied to integrated stochastic differential equations. In subsequent analysis we model the meso scale drift in two ways. The first parallels that of [10,11] with the Lagrangian velocity α-stable Lévy. The second and novel approach is to model the Lagrangian mesoscale acceleration as α-stable Lévy.

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0021-9991/$ - see front matter
doi:10.1016/j.jcp.2006.01.027

Lévy motions are ubiquitous in physics [15] and it is known that the transition densities, f, for α-stable Lévy and operator-stable Lévy motions satisfy the following Fokker–Planck equations [8],

$$\frac{\partial f}{\partial t} = -\mathbf{v}\cdot\nabla f + D\nabla^{\alpha} f \tag{1}$$

and

$$\frac{\partial f}{\partial t} = -\mathbf{v}\cdot\nabla f + \left(\frac{1}{2}\nabla\cdot\mathscr{A}\cdot\nabla + \mathscr{F}\right)f, \tag{2}$$

respectively. Here $\mathbf{v}$ is a divergence-free convective velocity, $\mathscr{A}$ is a symmetric positive definite $d\times d$ matrix of diffusion coefficients, $D = -1/\cos(\pi\alpha/2)$ describes the spreading rate of the dispersion, ∇^{α} is the fractional Laplacian:

$$(\widehat{\nabla^{\alpha} f})(\mathbf{k}) = \left[\int_{S_d} (-\mathrm{i}\mathbf{k}\cdot\mathbf{s})^{\alpha} M_v(\mathrm{d}\mathbf{s})\right]\hat{f}, \tag{3}$$

where $\mathbf{k}\cdot\mathbf{s}$ is the inner product and M_v is a finite measure on $S_d = \{\mathbf{s}\in\mathbf{R}^d : |\mathbf{s}| = 1\}$. The generalized fractional derivative $\mathscr{F}f$ is the inverse Fourier transform of

$$\widehat{\mathscr{F}f}(\mathbf{k}) \equiv \widehat{B}\hat{f}(\mathbf{k}) = \left(\int_{\mathbf{R}^d}\left(\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{y}} - 1 - \mathrm{i}\mathbf{k}\cdot\mathbf{y}\right)v(\mathrm{d}\mathbf{y})\right)\hat{f}(\mathbf{k}), \tag{4}$$

where v is the Lévy measure [4,9,14] and $\widehat{B}$ is a wavevector-dependent operator. The main difference between (1) and (2) is that in the former equation the order of the fractional derivative is the same in each direction, while in the later it changes with direction and it is this latter character that allows one to model swimming in preferential directions.

Many natural porous media exhibit a fractal character over some range of scales [1,12,16]. This fractal structure can induce fractal Lagrangian velocities or accelerations for conservative particles. Since an α-stable Lévy motion is a fractal with divider dimension α, we may without loss of generality model the drift velocity or acceleration in a fractal medium as a Lévy process. The degree of stability can be obtained experimentally by using particle tracking velocimetry (PTV) in conjunction with the finite-size Lyapunov exponent (FSLE) [6].

In subsequent sections we (i) review α-stable and operator-stable Lévy motions; (ii) assume a microbes behavior at the microscale (pore scale) is governed by a stochastic ordinary differential equation with stationary, ergodic, Markov drift and operator-stable Lévy diffusion; (iii) upscale the microscale equation to the mesoscale via a generalized central limit theorem (CLT); (iv) assume the drift on the mesoscale is α-stable Lévy with diffusion determined by the microscale asymptotics and again upscale via another CLT to the macroscale; (v) assume the acceleration on the mesoscale is α-stable Lévy with diffusion determined by the microscale asymptotics and again upscale via a CLT to the macroscale. The paper concludes with a brief summary of the results.

2. α-Stable Lévy and operator-stable Lévy processes

Consider the following integrated stochastic ordinary differential equation (SODE) in three dimensions:

$$\mathbf{X}(t) = \mathbf{X}(0) + \int_0^t \mathbf{V}(r)\,\mathrm{d}r + \rho\mathbf{L}(t), \quad t \geqslant 0, \tag{5}$$

where the stochastic process $\{\mathbf{V}(r)\}$ is α-stable Lévy, ρ is a constant and $\rho\mathbf{L}(t)$ is an operator-stable Lévy process [8]. Let $\widetilde{\mathbf{V}}(t) = \mathbf{V}(t) - \mathbf{V}(0)$ and $\widetilde{\mathbf{X}}(t) = \mathbf{X}(t) - \mathbf{X}(0)$. For each t, $\widetilde{\mathbf{V}}(t)$ has an α-stable distribution [5,7,13] ($\widetilde{\mathbf{V}}(t) \sim S_{\alpha_v}(tM_v, t\boldsymbol{\mu}_v)$), whose characteristic function, ϕ_v, is

$$\phi_v(t,\mathbf{k}) = \exp\left[-t\int_{S_d}|\mathbf{k}\cdot\mathbf{s}|^{\alpha_v}\left(1 - \mathrm{i}sgn(\mathbf{k}\cdot\mathbf{s})\tan\left(\frac{\alpha_v\pi}{2}\right)\right)M_v(\mathrm{d}\mathbf{s}) + \mathrm{i}t\mathbf{k}\cdot\boldsymbol{\mu}_v)\right], \tag{6}$$

where $0 < \alpha_v \leqslant 2$, $\alpha_v \neq 1$ and $sgn(x) = 1$ if $x > 0$, 0 if $x = 0$ and -1 otherwise and $\boldsymbol{\mu}_v$ is the mean tensor for the α-stable distribution. When $\alpha_v = 1$

$$\phi_v(t,\mathbf{k}) = \exp\left[-t\int_{S_d}|\mathbf{k}\cdot\mathbf{s}|\left(1+\mathrm{i}\frac{2}{\pi}sgn(\mathbf{k}\cdot\mathbf{s})\log|\mathbf{k}\cdot\mathbf{s}|\right)M_v(\mathrm{d}\mathbf{s})+\mathrm{i}t\mathbf{k}\cdot\boldsymbol{\mu}_v)\right]. \tag{7}$$

Henceforth, for simplicity in presentation we assume $\alpha_v \neq 1$. Let

$$\mathbf{Y}(t) = \int_0^t \mathbf{V}(r)\,\mathrm{d}r,$$

and decompose $\mathbf{Y}(t)$ as

$$\mathbf{Y}(t) = \int_0^t (\mathbf{V}(0)+\widetilde{\mathbf{V}}(r))\mathrm{d}r = t\mathbf{V}(0)+\widetilde{\mathbf{Y}}(t),$$

where $\widetilde{\mathbf{Y}}(t) = \int_0^t \widetilde{\mathbf{V}}(r)\,\mathrm{d}r$.

For each t, the characteristic function, ϕ_y, of $\widetilde{\mathbf{Y}}(t)$ is

$$\phi_y(t,\mathbf{k}) = \exp\left[-\frac{t^{1+\alpha_v}}{1+\alpha_v}\int_{S_d}|\mathbf{k}\cdot\mathbf{s}|^{\alpha_v}\left(1-\mathrm{i}sgn(\mathbf{k}\cdot\mathbf{s})\tan\left(\frac{\pi\alpha_v}{2}\right)\right)M_v(\mathrm{d}\mathbf{s})+\frac{\mathrm{i}t^2}{2}\mathbf{k}\cdot\boldsymbol{\mu}_v\right]. \tag{8}$$

The proof is given in [10].

3. Upscaling from microscale to mesoscale

On the microscale we assume the behavior of microbes depends on direction and so the integrated SODE is of the form

$$\mathbf{X}^{(0)}(t) = \mathbf{X}^{(0)}(0)+\int_0^t \mathbf{V}^{(0)}(r)\,\mathrm{d}r+\rho^{(0)}\mathbf{L}^{(0)}(t), \quad t \geqslant 0, \tag{9}$$

where $\mathbf{V}^{(0)}(r)$ is assumed stationary, ergodic and Markovian with the mean $\overline{\mathbf{V}}^{(0)}$, and $\rho^{(0)}\mathbf{L}^{(0)}(t)$ is an operator-stable Lévy process. Here, the superscript (0) means microscale. (We use (1) for mesoscale and (2) for macroscale.) The characteristic function, ϕ_ℓ, of $\rho^{(0)}\mathbf{L}^{(0)}(t)$ is [8]

$$\phi_\ell(t,\mathbf{k}) = \exp[\mathrm{i}t\mathbf{k}\cdot\boldsymbol{\mu}_\ell - t\mathbf{k}\cdot\mathscr{A}\cdot\mathbf{k}+t\widehat{B}(\mathbf{k})], \tag{10}$$

where $t\boldsymbol{\mu}_\ell^{(0)} = E[\mathbf{L}^{(0)}(t)]$ is the expected value of $\rho^{(0)}\mathbf{L}^{(0)}(t)$.

By using the classical central limit theorem it has been shown [3] that as $\lambda\to\infty$,

$$\lambda^{-1/2}\int_0^{\lambda t}\left(\mathbf{V}^{(0)}(r)-\overline{\mathbf{V}}^{(0)}\right)\mathrm{d}r \xrightarrow{d} \mathbf{B}^{(1)}(t), \tag{11}$$

a Brownian motion, where $\xrightarrow{d}$ means the convergence in distribution.

Let $\mathbf{Z}_\lambda^{(0)}(t) = (\rho^{(0)}\mathbf{L}^{(0)}(\lambda t)-\lambda t\boldsymbol{\mu}_\ell^{(0)})/\lambda^{1/2}$. Then the natural logarithm of the characteristic function, ϕ_λ, of $\mathbf{Z}_\lambda^{(0)}(t)$ is

$$\begin{aligned}\ln\phi_\lambda(t,\mathbf{k}) &= \ln E\left[\exp(\mathrm{i}\mathbf{k}\cdot\mathbf{Z}_\lambda^{(0)}(t))\right] = -t\mathbf{k}\cdot\mathscr{A}\cdot\mathbf{k}+t\int_{\mathbf{R}^d}\lambda\left(\mathrm{e}^{\mathrm{i}\lambda^{-1/2}\mathbf{k}\cdot\boldsymbol{\ell}}-1-\mathrm{i}\lambda^{-1/2}\mathbf{k}\cdot\boldsymbol{\ell}\right)\nu(\mathrm{d}\boldsymbol{\ell})\\ &= -t\mathbf{k}\cdot\mathscr{A}\cdot\mathbf{k}+t\int_{\mathbf{R}^d}\lambda(\cos(\lambda^{-1/2}\mathbf{k}\cdot\boldsymbol{\ell})-1)\nu(\mathrm{d}\boldsymbol{\ell})+\mathrm{i}t\int_{\mathbf{R}^d}\lambda(\sin(\lambda^{-1/2}\mathbf{k}\cdot\boldsymbol{\ell})-\lambda^{-1/2}\mathbf{k}\cdot\boldsymbol{\ell})\nu(\mathrm{d}\boldsymbol{\ell}).\end{aligned}$$

Observe that the integrands of the two integrals are monotonically increasing as $\lambda\to\infty$. By the monotone convergence theorem, the two sequences of integrals converge. Therefore, for each $\mathbf{k}$

$$\ln\phi_\lambda(t,\mathbf{k}) \to \left(-t\mathbf{k}\cdot\mathscr{A}\cdot\mathbf{k}-t\int_{\mathbf{R}^d}\frac{1}{2}(\mathbf{k}\cdot\boldsymbol{\ell})^2\nu(\mathrm{d}\boldsymbol{\ell})\right), \quad \lambda\to\infty.$$

Hence,

$$\mathbf{Z}_\lambda^{(0)}(t) \xrightarrow{d} \mathbf{Z}^{(1)}(t), \quad \lambda\to\infty \tag{12}$$

whose characteristic function, ϕ_z, is

$$\phi_z(t,\mathbf{k}) = \exp\left[-t\mathbf{k}\cdot\mathscr{A}\cdot\mathbf{k} - \frac{t}{2}\int_{\mathbf{R}^d}(\mathbf{k}\cdot\boldsymbol{\ell})^2 v(\mathrm{d}\boldsymbol{\ell})\right]. \tag{13}$$

In order to use (11) and (12), we look at

$$\frac{\widetilde{\mathbf{X}}^{(0)}(\lambda t) - \lambda t(\overline{\mathbf{V}}^{(0)} + \boldsymbol{\mu}_\ell^{(0)})}{\lambda^{1/2}} = \frac{\mathbf{Y}(\lambda t) - \lambda t\overline{\mathbf{V}}^{(0)}}{\lambda^{1/2}} + \frac{\rho\mathbf{L}(\lambda t) - \lambda t\boldsymbol{\mu}_\ell^{(0)}}{\lambda^{1/2}}.$$

Eqs. (11) and (12) then show convergence in distribution:

$$\frac{\widetilde{\mathbf{X}}^{(0)}(\lambda t) - \lambda t(\overline{\mathbf{V}}^{(0)} + \boldsymbol{\mu}_\ell^{(0)})}{\lambda^{1/2}} \xrightarrow{d} (\mathbf{B}^{(1)}(t) + \mathbf{Z}^{(1)}(t)) \equiv \rho^{(1)}\widetilde{\mathbf{L}}^{(1)}(t), \tag{14}$$

whose characteristic function, ψ_ℓ, is

$$\psi_\ell(t,\mathbf{k}) = \exp\left[-t\int_{S_d}|\mathbf{k}\cdot\mathbf{s})|^2 M_l(\mathrm{d}\mathbf{s}) - t\mathbf{k}\cdot\mathscr{A}\cdot\mathbf{k} - \frac{t}{2}\int_{\mathbf{R}^d}(\mathbf{k}\cdot\boldsymbol{\ell})^2 v(\mathrm{d}\boldsymbol{\ell})\right]. \tag{15}$$

4. Upscaling from mesoscale to macroscale with $\mathbf{V}^{(1)}(t)$ α_v-stable Lévy

Let $\mathbf{V}^{(1)}(t)$ be α_v-stable Lévy with $1 < \alpha_v \leqslant 2$. We assume that for each t, $\mathbf{V}^{(1)}(t)$ has constant mean and $\mathbf{V}^{(1)}(0)$ has an initial distribution π_v.

$$E(\mathbf{V}^{(1)}(t)) = E(\mathbf{V}^{(1)}(0)) = \overline{\mathbf{V}}^{(1)} \quad \text{for any } t \geqslant 0. \tag{16}$$

Since $E(\widetilde{\mathbf{V}}^{(1)}(t)) = \mathbf{0}$ for each time t, $\boldsymbol{\mu}_v = \mathbf{0}$. We also assume periodicity of $\mathbf{V}^{(1)}(t) \equiv \mathbf{V}^{(1)}(t, \boldsymbol{\xi}^{(1)}(0))$ in the initial data:

$$\mathbf{V}^{(1)}(t, \boldsymbol{\xi}^{(1)}(0) + \boldsymbol{v}^{(1)}) = \mathbf{V}^{(1)}(t, \boldsymbol{\xi}^{(1)}(0)),$$

where $\boldsymbol{v}^{(1)} = (v_1^{(1)}, \ldots, v_d^{(1)})$ for integers $v_k^{(1)}, k = 1, \ldots, d$ and $\boldsymbol{\xi}^{(1)}(\cdot) = \boldsymbol{\xi}^{(1)}(\cdot, \omega)$ is a random particle path in the porous medium with ω an elementary event. Let

$$[\mathbf{V}^{(1)}] = E\left[\frac{1}{|\Omega_0|}\int_{\Omega_0}\frac{1}{|B|}\int_B \mathbf{V}^{(1)}(t, \boldsymbol{\xi}^{(1)}(0))\,\mathrm{d}t\,\mathrm{d}\boldsymbol{\xi}^{(1)}(0)\right], \tag{17}$$

where $\Omega_0 = \Omega_0(\omega) = \{\boldsymbol{\xi}^{(1)}(0,\omega): \boldsymbol{\xi}^{(1)}(0,\omega) \in [0,1]^d\}$ and $B = B(\omega) = \{t \geqslant 0: \boldsymbol{\xi}^{(1)}(t,\omega) \in [0,1]^d\}$. Set $\boldsymbol{\mu}_\ell^{(1)} = \overline{\mathbf{V}}^{(0)} + \boldsymbol{\mu}_\ell^{(0)}$ and $\rho^{(1)}\mathbf{L}^{(1)}(t) = \boldsymbol{\mu}_\ell^{(1)} + \rho^{(1)}\widetilde{\mathbf{L}}^{(1)}(t)$.

For $t = \lambda t'$, we showed the convergence [10]:

$$\frac{\mathbf{Y}^{(1)}(\lambda t') - \lambda t'\overline{\mathbf{V}}^{(1)}}{\lambda^{1+1/\alpha_v}} \xrightarrow{d} \mathbf{W}(t'), \quad \lambda \to \infty.$$

The characteristic function, ψ_w, of $\mathbf{W}(t')$ is

$$\psi_w(t',\mathbf{k}) = \exp\left[-\frac{t'^{1+\alpha_v}}{1+\alpha_v}\int_{S_d}|\mathbf{k}\cdot\mathbf{s}|^{\alpha_v}\left(1 - \mathrm{i}sgn(\mathbf{k}\cdot\mathbf{s})\tan\left(\frac{\pi\alpha_v}{2}\right)\right)M_v(\mathrm{d}\mathbf{s})\right]. \tag{18}$$

Let $\widetilde{\mathbf{Y}}^{(2)}(t') = \lim_{\lambda\to\infty}\widetilde{\mathbf{Y}}^{(1)}(\lambda t')$. Thus, for each $t > 0$ we have

$$\mathbf{Y}^{(2)}(t) \approx t\overline{\mathbf{V}}^{(1)} + \widetilde{\mathbf{Y}}^{(2)}(t). \tag{19}$$

To obtain a central limit theorem for $\widetilde{\mathbf{X}}^{(1)}(t)$, let

$$\begin{aligned}\mathbf{Z}_\lambda^{(1)}(t') &\equiv \frac{\widetilde{\mathbf{X}}^{(1)}(\lambda t') - ([\mathbf{V}^{(1)}] + \boldsymbol{\mu}_\ell^{(1)})\lambda t'}{\lambda^{1+1/\alpha_v}} \\ &= \frac{t'(\overline{\mathbf{V}}^{(1)} - [\mathbf{V}^{(1)}])}{\lambda^{1/\alpha_v}} + \frac{\mathbf{Y}^{(1)}(\lambda t') - \lambda t'\overline{\mathbf{V}}^{(1)}}{\lambda^{1+1/\alpha_v}} + \frac{\rho^{(1)}\mathbf{L}^{(1)}(\lambda t') - \lambda t'\boldsymbol{\mu}_\ell^{(1)}}{\lambda^{1+1/\alpha_v}}.\end{aligned} \tag{20}$$

As $\lambda \to \infty$, the first and second terms of the right hand side of (20) converge to zero and $\widetilde{\mathbf{Y}}^{(2)}(t')$ in distribution, respectively. To obtain the convergence of the last term of the right hand side of (20), we consider the natural logarithm of its characteristic function:

$$\ln E\left[\exp\left(\mathrm{i}\lambda^{-1-1/\alpha_v}\mathbf{k}\cdot\left(\rho^{(1)}\mathbf{L}^{(1)}(\lambda t') - \lambda t'\boldsymbol{\mu}_\ell^{(1)}\right)\right)\right]$$
$$= t'\int_{\mathbf{R}^d}\lambda(\mathrm{e}^{\mathrm{i}\lambda^{-1-1/\alpha_v}\mathbf{k}\cdot\boldsymbol{\ell}} - 1 - \mathrm{i}\lambda^{-1-1/\alpha_v}\mathbf{k}\cdot\boldsymbol{\ell})v(\mathrm{d}\boldsymbol{\ell}) - t'\lambda^{-1-2/\alpha_v}\mathbf{k}\cdot\mathscr{A}\cdot\mathbf{k} \tag{21}$$

By the monotone convergence theorem, the right hand side of (21) converges to zero as $\lambda \to \infty$. So, $\mathbf{Z}_\lambda^{(1)}(t')$ converges to $\widetilde{\mathbf{Y}}^{(2)}(t')$ in distribution. Thus, for each $t > 0$, we can approximate the stochastic process $\widetilde{\mathbf{X}}^{(2)}(t) = \lim_{\lambda\to\infty}\widetilde{\mathbf{X}}^{(1)}(\lambda t)$ as

$$\widetilde{\mathbf{X}}^{(2)}(t) \approx t\left([\mathbf{V}^{(1)}] + \boldsymbol{\mu}_\ell^{(1)}\right) + \widetilde{\mathbf{Y}}^{(2)}(t). \tag{22}$$

It is possible to derive the Fokker–Planck (Advection–Dispersion) equation [10] that the transition density $f(t,\mathbf{x})$ for $\widetilde{\mathbf{X}}^{(2)}(t)$ satisfies:

$$\frac{\partial f}{\partial t} = -\mathbf{v}^{(2)}\cdot\nabla f + D^{(2)}(t)\nabla^{\alpha_v} f, \tag{23}$$

where $\mathbf{v}^{(2)} = [\mathbf{V}^{(1)}] + \boldsymbol{\mu}_\ell^{(1)}$, $D^{(2)}(t) = -t^{\alpha_v}/\cos(\pi\alpha_v/2)$ and $\nabla^\alpha f$ is defined by

$$(\widehat{\nabla^\alpha f})(t,\mathbf{k}) = \left[\int_{S_d}(-\mathrm{i}\mathbf{k}\cdot\mathbf{s})^\alpha M_v(\mathrm{d}\mathbf{s})\right]\hat{f}(t,\mathbf{k}). \tag{24}$$

5. Upscaling from mesoscale to macroscale with α-stable Lévy acceleration

Next, we consider the Lagrangian acceleration to be fractal on the mesoscale rather than the velocity which is the integral of the acceleration. As with the Lagrangian velocity, we assume that the Lagrangian acceleration, $\mathbf{A}^{(1)}$, on the mesoscale is α_a-stable Lévy ($\mathbf{A}^{(1)} \sim S_{\alpha_a}(tM_a, t\boldsymbol{\mu}_a)$) with $1 < \alpha_a \leqslant 2$ and constant mean, $E(\mathbf{A}^{(1)}(t)) = E(\mathbf{A}^{(1)}(0)) = \overline{\mathbf{A}}^{(1)}$ for each $t > 0$. Let

$$\mathbf{Y}^{(1)}(t) = \int_0^t\int_0^u \mathbf{A}^{(1)}(r)\,\mathrm{d}r\,\mathrm{d}u.$$

We decompose $\mathbf{Y}^{(1)}(t)$ as follows:

$$\mathbf{Y}^{(1)}(t) = \int_0^t\int_0^u\left(\mathbf{A}^{(1)}(0) + \widetilde{\mathbf{A}}^{(1)}(r)\right)\mathrm{d}r\,\mathrm{d}u = \frac{t^2}{2}\mathbf{A}^{(1)}(0) + \widetilde{\mathbf{Y}}^{(1)}(t),$$

where

$$\widetilde{\mathbf{Y}}^{(1)}(t) = \int_0^t\int_0^u \widetilde{\mathbf{A}}^{(1)}(r)\,\mathrm{d}r\,\mathrm{d}u \tag{25}$$

and $\boldsymbol{\mu}_a = E(\widetilde{\mathbf{A}}^{(1)}(r)) = \mathbf{0}$. By switching the two integrals in (25), we obtain

$$\widetilde{\mathbf{Y}}^{(1)}(t) = \int_0^t (t-r)\widetilde{\mathbf{A}}^{(1)}(r)\,\mathrm{d}r.$$

For each t, the characteristic function, ϕ_y, of $\widetilde{\mathbf{Y}}(t)$ is

$$\phi_y(t,\mathbf{k}) = \exp\left[-\frac{t^{1+2\alpha_a}}{2^{\alpha_a}(1+2\alpha_a)}\int_{S_3}|\mathbf{k}\cdot\mathbf{s}|^{\alpha_a}\psi(\mathbf{k},\mathbf{s},\alpha_a)M_a(\mathrm{d}\mathbf{s})\right], \tag{26}$$

where $\psi(\mathbf{k}, \mathbf{s}, \alpha_a) = 1 - \mathrm{i}\,sgn(\mathbf{k}\cdot\mathbf{s})\tan(\pi\alpha_a/2)$. We can prove (26) in a fashion similar to that in [10].

We take the same assumption on the microscale as in the previous section. and upscale from mesoscale to macroscale. We assume periodicity of $\mathbf{A}^{(1)}(r) \equiv \mathbf{A}^{(1)}(r, \boldsymbol{\xi}^{(1)}(0))$ in the initial data:

$$\mathbf{A}^{(1)}(r, \boldsymbol{\xi}^{(1)}(0) + \boldsymbol{v}^{(1)}) = \mathbf{A}^{(1)}(r, \boldsymbol{\xi}^{(1)}(0)),$$

where $\boldsymbol{v}^{(1)} = (v_1^{(1)}, \ldots, v_d^{(1)})$ for integers $v_k^{(1)}, k = 1, \ldots, d$ and $\boldsymbol{\xi}^{(1)}(\cdot) = \boldsymbol{\xi}^{(1)}(\cdot,\omega)$ is a particle path in the porous medium. Let

$$[\mathbf{A}^{(1)}] = E\left[\frac{1}{|\Omega_0|}\int_{\Omega_0}\frac{1}{|B|}\int_B \mathbf{A}^{(1)}(t, \boldsymbol{\xi}^{(1)}(0))\,\mathrm{d}t\,\mathrm{d}\boldsymbol{\xi}^{(1)}(0)\right], \tag{27}$$

where $\Omega_0 = \Omega_0(\omega) = \{\boldsymbol{\xi}^{(1)}(0,\omega)\colon \boldsymbol{\xi}^{(1)}(0,\omega) \in [0,1]^d\}$ and $B = B(\omega) = \{t \geqslant 0\colon \boldsymbol{\xi}^{(1)}(t,\omega) \in [0,1]^d\}$. Set $\rho^{(1)}\mathbf{L}^{(1)}(t) = t\boldsymbol{\mu}_\ell^{(1)} + \rho^{(1)}\widetilde{\mathbf{L}}^{(1)}(t)$.

For $t = \lambda t'$, we can show the convergence by the same argument as in [10]:

$$\frac{\mathbf{Y}^{(1)}(\lambda t') - \frac{1}{2}(\lambda t')^2\overline{\mathbf{A}}^{(1)}}{\lambda^{2+1/\alpha_a}} \xrightarrow{d} \mathscr{W}(t'), \quad \lambda \to \infty.$$

The characteristic function, ϕ_w, of $\mathscr{W}(t')$ is the same as (26). Let $\widetilde{\mathbf{Y}}^{(2)}(t') = \lim_{\lambda\to\infty}\widetilde{\mathbf{Y}}^{(1)}(\lambda t')$. Thus, for each $t > 0$ we have

$$\mathbf{Y}^{(2)}(t) \approx \frac{t^2}{2}\overline{\mathbf{A}}^{(1)} + \widetilde{\mathbf{Y}}^{(2)}(t). \tag{28}$$

To obtain a central limit theorem for $\widetilde{\mathbf{X}}^{(1)}(t)$, let

$$\begin{aligned}\mathscr{Z}_\lambda^{(1)}(t') &\equiv \frac{\widetilde{\mathbf{X}}^{(1)}(\lambda t') - \frac{(\lambda t')^2}{2}[\mathbf{A}^{(1)}] - \lambda t'\boldsymbol{\mu}_\ell^{(1)}}{\lambda^{2+1/\alpha_a}} \\ &= \frac{(t')^2(\overline{\mathbf{A}}^{(1)} - [\mathbf{A}^{(1)}])}{2\lambda^{1/\alpha_a}} + \frac{\mathbf{Y}^{(1)}(\lambda t') - \frac{1}{2}(\lambda t')^2\overline{\mathbf{A}}^{(1)}}{\lambda^{2+1/\alpha_a}} + \frac{\rho^{(1)}\mathbf{L}^{(1)}(\lambda t') - \lambda t'\boldsymbol{\mu}_\ell^{(1)}}{\lambda^{2+1/\alpha_a}}.\end{aligned} \tag{29}$$

As $\lambda \to \infty$, the first and second terms of the right hand side of (29) converge to zero and $\widetilde{\mathbf{Y}}^{(2)}(t')$ in distribution, respectively. By the monotone convergence theorem, the last term of the right hand side of (29) converges to zero in distribution as $\lambda \to \infty$. So, $\mathscr{Z}_\lambda^{(1)}(t')$ converges to $\widetilde{\mathbf{Y}}^{(2)}(t')$ in distribution. Thus, for each $t > 0$, we can approximate the stochastic process $\widetilde{\mathbf{X}}^{(2)}(t) = \lim_{\lambda\to\infty}\widetilde{\mathbf{X}}^{(1)}(\lambda t)$ as

$$\widetilde{\mathbf{X}}^{(2)}(t) \approx \left(\frac{t^2}{2}[\mathbf{A}^{(1)}] + t\boldsymbol{\mu}_\ell^{(1)}\right) + \widetilde{\mathbf{Y}}^{(2)}(t). \tag{30}$$

It is possible to derive the Fokker–Planck (Advection–Dispersion) equation [10] that the transition density $f(t,\mathbf{x})$ for $\widetilde{\mathbf{X}}^{(2)}(t)$ satisfies:

$$\frac{\partial f}{\partial t} = -\mathbf{v}^{(2)}\cdot\nabla f + D^{(2)}(t)\nabla^{\alpha_a} f, \tag{31}$$

where $\mathbf{v}^{(2)} = t[\mathbf{A}^{(1)}]/2 + \boldsymbol{\mu}_\ell^{(1)}$ and $D^{(2)}(t) = -t^{2\alpha_a}/\cos(\pi\alpha_a/2)$.

6. Summary

We have been studying the movement of motile microbes in porous media for a number of years. This manuscript represents our latest attempt to capture, on multiple scales, the ability of a microbe to move in a preferential direction toward an energy source. The preferential movement is governed by an operator-stable Lévy motion at the pore (micro) scale. At this scale, the particle also experiences a convective drift velocity that is stationary, ergodic and Markovian. The integrated SODE governing the particle transport is upscaled to the Darcy (meso) scale via a generalized central limt theorem. At the Darcy scale diffusion is governed by the asymptotics of the pore scale and drift is governed by an α_v-stable Lévy velocity process or an α_a-stable Lévy acceleration. The velocity/acceleration process was chosen to be Lévy to represent the fractal character of the velocity/acceleration at the Darcy scale. The mesoscale integrated SODE was upscaled to the field (macro) scale via application of a second central limit theorem. Fractional advective-dispersion equations with time-dependent dispersion coefficients were presented at the macro scale.

In addition to its application to motile microbe transport in fractal porous media, the work presented has application to other colloids and fractal biological media such as aerosols in lung tissue [16].

Acknowledgement

The work reported herein has been sponsored by the National Science Foundation under Contract Nos. 0310029-EAR, 0417555-EAR and 0003878-EAR.

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